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Governing equations

(under the hydrostatic Boussinesq and incompressibility approximations)

See also Pinardi's notes

Reference coordinate system

Unless otherwise stated the reference system will be the following

Cushman-Roisin (1992)

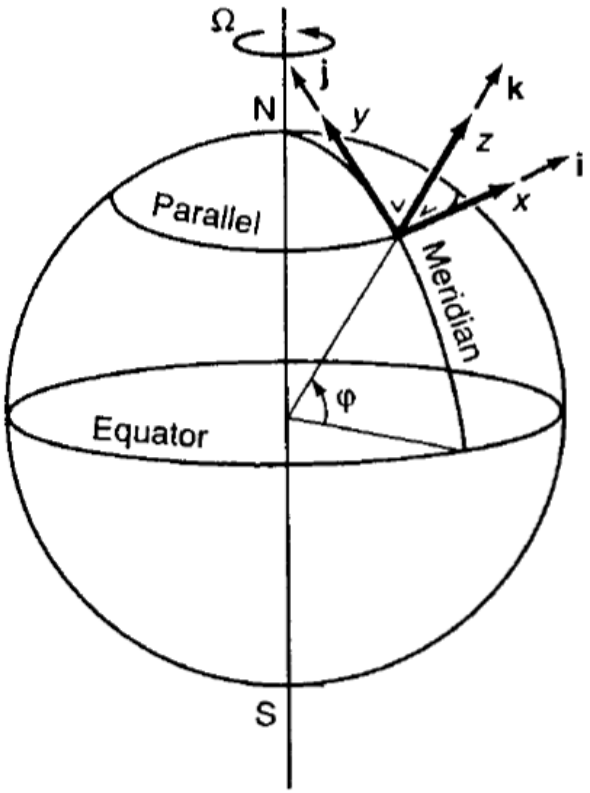


Figure 2-8 Definition of the local Cartesian framework of reference on a spherical earth. The coordinate x is directed eastward, y is northward, and z is upward.

Equation of motion

$$\frac{\partial u}{\partial t}+\vec{u}\cdot\nabla u-fv+f^{*}w=-\frac{1}{\rho_0}\frac{\partial p}{\partial x}+\frac{A_H}{\rho_0}\nabla_2^2 u+\frac{A_v}{\rho_0}\frac{\partial^2 u}{\partial z^2}$$

$$\frac{\partial v}{\partial t}+\vec{u}\cdot\nabla v+fu=-\frac{1}{\rho_0}\frac{\partial p}{\partial y}+\frac{A_H}{\rho_0}\nabla_2^2 v+\frac{A_v}{\rho_0}\frac{\partial^2 v}{\partial z^2}$$

$$\frac{\partial w}{\partial t}+\vec{u}\cdot\nabla w-fu=-\frac{1}{\rho_0}\frac{\partial p}{\partial z}-g\frac{\rho}{\rho_0}+\frac{A_H}{\rho_0}\nabla_2^2 w+\frac{A_v}{\rho_0}\frac{\partial^2 w}{\partial z^2}$$

$\nabla_2^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial x^2}$ Bilaplacian Horizontal operator $\vec{u}=(u,v,w)$ 3D velocity vector p pressure

$f=2\Omega\sin\theta; f^{*}=2\Omega\cos\theta$ Coriolis terms with $\theta =$ Latitude and $\Omega=7.29\cdot 10^{-5}s^{-1}$ Earth angular vel.

$\vec{u}\cdot\nabla=u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}$ Advective term $\rho;\rho_0 =$ Density and reference density

$A_H;A_v$ Viscosity coefficients[Kg/(ms)] (include turbulent and molecular viscosity) $g =$ gravity

Continuity, Hydrostatic, State Eq.'s

$$\nabla \cdot \vec{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Continuity equation

$$\frac{\partial p}{\partial z} = -\rho g$$

Hydrostatic equation

$$\rho = \rho\left(T, p, S\right)$$

Complex polynomial of Temperature (T), pressure And salinity (S).
UNESCO Equation of state

Temperature and Salinity
(Conservative tracers)

$$\frac{\partial(T,S)}{\partial t}+\vec{u}\cdot\nabla(T,S)=K_H\nabla_2^2(T,S)+K_v\frac{\partial^2(T,S)}{\partial z^2}$$

K_H, K_v [m^2/s]Horizontal and vertical diffusivities (include molecular diffusivity)

Cinematic Boundary Conditions

Horizontal Boundaries (a coast):

The "no-slip" condition.
The normal velocity field is nil, i.e.: $\vec{u}_H \cdot \vec{n} = 0$ Where $\vec{n}$ is the normal to the boundary
For a viscous fluid it must be imposed also a nil parallel (to the boundary) Velocity. Then:

$$\vec{u}_H \times \vec{n} = 0$$

Conservative Tracers (T and S)

The normal (to the boundary) component of the diffusive flux $J_{diff} = K\vec{\nabla}(T,S)$ is
Imposed to be nil: $J_{diff} \cdot \vec{n} = 0$

Cinematic Boundary Conditions

Vertical Boundaries (Ocean surface and bottom):

If a “Free surface” (η) is adopted we have:

$$w\big|_{z=\eta} = \frac{D\eta}{Dt}$$ (The particle does not leave the free surface)

At the ocean bottom [$z=-H(x,y)$]:

$$w\big|_{z=-H} = -u \cdot \nabla H$$ (the particle does not leave the bathymetry)

Dynamic Boundary Conditions

Equation of motion

At the Free surface ($z=\eta$) we have:

$$A_v \left.\frac{\partial \vec{u}}{\partial z}\right|_{z=\eta} = \vec{\tau}_w \quad \text{where:}$$

$$\tau_w^{(x)} = \rho_a C_D |\vec{u}_w| u_w \quad \text{and} \quad \tau_w^{(y)} = \rho_a C_D |\vec{u}_w| v_w$$ Is the wind-stress

ρ_a air density, C_D drag coefficient $\vec{u}_w$ wind velocity

At the ocean bottom ($z=-H$) we have:

$$A_v \left.\frac{\partial \vec{u}}{\partial z}\right|_{z=-H} = \vec{\tau}_b \quad \text{where:}$$

$$\tau_b^{(x)} = \rho_0 C_b |\vec{u}_b| u_b \quad \text{and} \quad \tau_w^{(y)} = \rho_a C_D |\vec{u}_w| v_w$$ is the bottom stress

ρ_0 reference seawater density C_b drag coefficient $\vec{u}_b$ current velocity near the bottom

REMEMBER $u_* = \sqrt{\dfrac{\tau_w}{\rho_0}}$ "Friction velocity"

Dynamic Boundary Conditions

Temperature equation

At the Free surface ($z=\eta$) we have:

$$K_v \left.\frac{\partial T}{\partial z}\right|_{z=\eta} = \frac{Q}{C_p \rho_0}$$ where:

C_p Specific heat $\quad Q = Q_S - Q_B - Q_E - Q_H$ Heat flux at ocean surface with:

Q_S Incoming solar Radiation $\quad Q_B$ net infra-red flux $\quad Q_E$ Latent heat flux

Q_H Sensible heat flux

At the bottom the boundary condition is adiabatic

$$K_v \left.\frac{\partial T}{\partial z}\right|_{z=-H} = 0$$

Dynamic Boundary Conditions

Surface fresh water flux

The surface ($z=\eta$) fresh water flux is defined as:

$$W = E - P - \frac{R}{A}$$ where:

E and P Evaporation and precipitation [m/s]

R River runoff [m^3/s] A Area trough which the runoff occurs [m^2]

and the surface boundary condition for w (seen before) is modified as follows:

$$w\big|_{z=\eta} - \frac{D\eta}{Dt} = W$$

Dynamic Boundary Conditions

Salinity equation

The definition of a fresh water surface flux implies a modification (dilution/concentration) Of the surface salinity values. The modification is accounted for by the following Boundary condition for salinity:

$$K_v \left.\frac{\partial S}{\partial z}\right|_{z=\eta} = W\, S\big|_{z=\eta}$$

It is implied that the incoming/outgoing water has salinity =0.

At the bottom we have obviously:

$$K_v \left.\frac{\partial S}{\partial z}\right|_{z=-H} = 0$$

Shallow water equations

Homogeneous, Hydrostatic and incompressible fluid with:

$$\frac{D}{L} << 1$$

D= Vertical scale, *L*= Horizontal scale

Here we will use the linearised equations (assuming then the local term much larger than the Advective terms), but (differently from the Physical oceanography notes) the vertical diffusion term is retained

NB: $\frac{1}{\rho_0}\frac{\partial p}{\partial(x,y)}$ under homogeneous conditions becomes $g\frac{\partial \eta}{\partial(x,y)}$

Therefore we have:

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x} + \frac{A_v}{\rho_0}\frac{\partial^2 u}{\partial z^2}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y} + \frac{A_v}{\rho_0}\frac{\partial^2 v}{\partial z^2}$$

Together with the continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial w}{\partial z}$$

Shallow water equations

Vertically integrating

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x} + \frac{A_v}{\rho_0}\frac{\partial^2 u}{\partial z^2} \qquad \frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y} + \frac{A_v}{\rho_0}\frac{\partial^2 v}{\partial z^2} \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial w}{\partial z}$$

With the boundary conditions described before and posing the resulting horizontal transport:

$$U = \int_{-H}^{\eta} u\,dz \qquad V = \int_{-H}^{\eta} v\,dz$$

We end with the "**Transport equations**":

$$\frac{\partial U}{\partial t} - fV = -gH\frac{\partial \eta}{\partial x} + \frac{1}{\rho_0}\left(\tau_w^x - \tau_b^x\right) \qquad \frac{\partial V}{\partial t} + fU = -gH\frac{\partial \eta}{\partial y} + \frac{1}{\rho_0}\left(\tau_w^y - \tau_b^y\right)$$

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = -\frac{\partial \eta}{\partial t}$$

The “Local” and the “Global” problem

The shallow water equations system :

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x} + \frac{A_v}{\rho_0}\frac{\partial^2 u}{\partial z^2} \qquad \frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y} + \frac{A_v}{\rho_0}\frac{\partial^2 v}{\partial z^2} \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial w}{\partial z}$$

Can be used to solve to so-called “local” problem i.e. determine the $u(z)$ and $v(z)$ profiles from a “local” calculations (given x, y) provided that the A_v distribution is known (or assumed) and the η Gradient is an external input.

The transport equations system:

$$\frac{\partial U}{\partial t} - fV = -gH\frac{\partial \eta}{\partial x} + \frac{1}{\rho_0}\left(\tau_w^x - \tau_b^x\right) \quad \frac{\partial V}{\partial t} + fU = -gH\frac{\partial \eta}{\partial y} + \frac{1}{\rho_0}\left(\tau_w^y - \tau_b^y\right) \quad \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = -\frac{\partial \eta}{\partial t}$$

Can calculate the Basin-wide (“global”) distribution of pressure (η) and transport (U, V).

τ_w is an external input, τ_b can be computed from the bottom velocities originating from the solution of the local problem.

The “Local” and the “Global” problem

N.B.: In the transport equation system $gH = c^2$

With $c = \sqrt{gH}$ being the free gravity wave phase velocity.
Then the transport equation system can be written also as

$$\frac{\partial U}{\partial t} - fV = -c^2 \frac{\partial \eta}{\partial x} + \frac{1}{\rho_0}\left(\tau_w^x - \tau_b^x\right) \qquad \frac{\partial V}{\partial t} + fU = -c^2 \frac{\partial \eta}{\partial y} + \frac{1}{\rho_0}\left(\tau_w^y - \tau_b^y\right)$$

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = -\frac{\partial \eta}{\partial t}$$

Vorticity and Potential Vorticity equations

Planetary vorticity: $f=2\Omega sin\theta$

Relative vorticity (vertical component): $\xi_z = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}$

Absolute vorticity: $(\xi_z + f)$

Potential vorticity: $\dfrac{\xi_z + f}{H+\eta}$

Starting from the linearised, vertically Integrated transport equations, by Cross derivation and subtraction, the Vertically integrated linearised analog of theVorticity equations is defined as follows:

$$\frac{D}{Dt}\left(\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}\right) = f\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}\left(\boldsymbol{\tau}_w^{(y)} - \boldsymbol{\tau}_b^{(y)}\right) - \frac{\partial}{\partial y}\left(\boldsymbol{\tau}_w^{(x)} - \boldsymbol{\tau}_b^{(x)}\right)$$

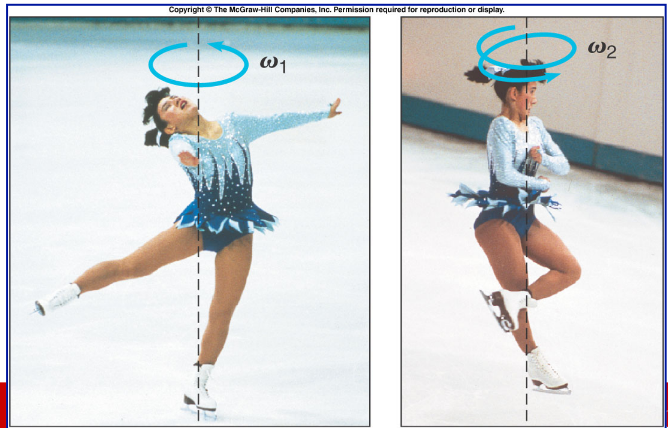
Vorticity and Potential Vorticity equations

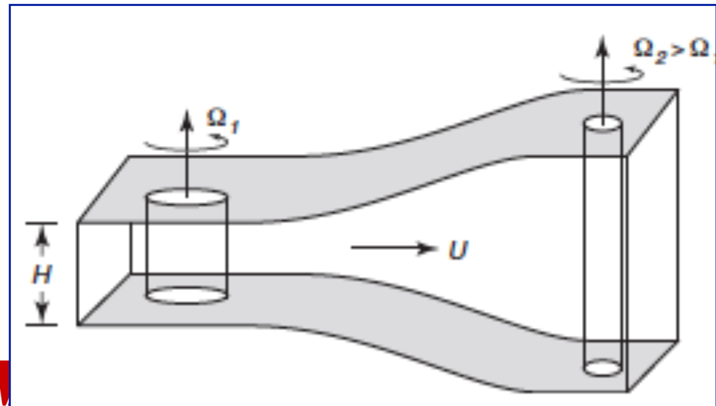
$$\frac{D}{Dt}\left(\frac{\xi_z + f}{H+\eta}\right) = \frac{1}{(H+\eta)}\left[\frac{\partial}{\partial x}\frac{1}{\rho_0}\left(\boldsymbol{\tau}_w^{(y)} - \boldsymbol{\tau}_b^{(y)}\right) - \frac{\partial}{\partial y}\frac{1}{\rho_0}\left(\boldsymbol{\tau}_w^{(x)} - \boldsymbol{\tau}_b^{(x)}\right)\right]$$

Potential vorticity (P.V.) originates by net the curl of the friction affecting the water column. When This vanishes P.V. is conserved (see eq. 7.19 in Pinardi's notes)

$$\frac{D}{Dt}\left(\frac{\xi_z + f}{H+\eta}\right) = 0$$

P.V. conservatipon require generating "cyclonic"/"anticyclonic vorticity or Water column stretching/squashing ("Skater" effect)





Vorticity and Potential Vorticity equations

Starting from the linearised, vertically Integrated transport equations, by Cross derivation and subtraction, the Vertically integrated linearised analog of theVorticity equations is defined as follows:

$$\frac{D}{Dt}\left(\frac{\partial V}{\partial x}-\frac{\partial U}{\partial y}\right)=f\frac{\partial \eta}{\partial t}+\frac{\partial}{\partial x}\left(\tau_w^{(y)}-\tau_b^{(y)}\right)-\frac{\partial}{\partial y}\left(\tau_w^{(x)}-\tau_b^{(x)}\right)$$

Rate of change of the Depth integrated relative Vorticity.

Vorticity generation Trough surface elevation Changes(squashing/stretching)

Curl of the surface-bottom Stress difference

Geostrophic Balance

Steady state, homogeneous conditions, no surface and bottom stress. The shallow water Equation reduces to the geostrophic relations:

$$-fv = -g\frac{\partial \eta}{\partial x} \qquad fu = -g\frac{\partial \eta}{\partial y} \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

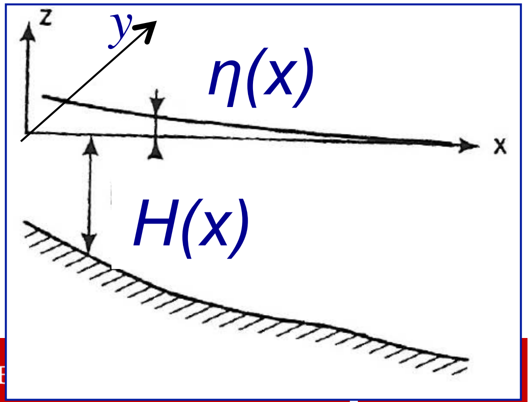
The corresponding transport equations are:

$$-fV = -gH\frac{\partial \eta}{\partial x} \qquad fU = -gH\frac{\partial \eta}{\partial y} \qquad \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0$$

Aligning the y axis with the direction of the transport vector, so that $U=0$ we have:

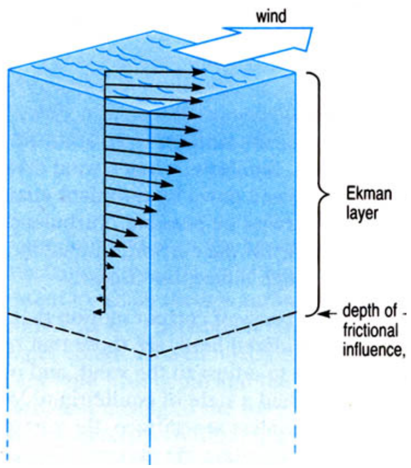
$$-fV = -gH\frac{\partial \eta}{\partial x} \qquad 0 = -gH\frac{\partial \eta}{\partial y} \qquad \frac{\partial V}{\partial y} = 0$$

Transport along isobaths η varies only in the cross-isobaths Directions (an elementary coastal flow model)



Ekman equations and transport

Assuming an indefinitely large and deep ocean, Steady state, homogeneous conditions.
The shallow water equations reduces to the "Ekman" equations (see Pinardi's notes eq's 5.3 a, b:



$$fu_e = \frac{A_v}{\rho_0}\frac{\partial^2 v}{\partial z^2}$$

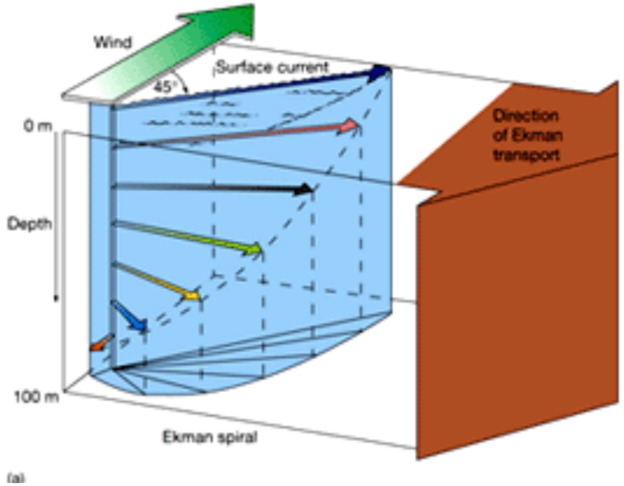
$$-fv_e = \frac{A_v}{\rho_0}\frac{\partial^2 u}{\partial z^2}$$

The "local" problem solution is the well known "Ekman spiral"

The corresponding transport equations (global problem) are (eq's 5.14 in Pinardi's notes):

$$U = \int_{-2D_e}^{\eta} \rho_0 u_e \, dz = \frac{\tau_w^{(y)}}{f}$$

$$-V = -\int_{-2D_e}^{\eta} \rho_0 v_e \, dz = \frac{\tau_w^{(x)}}{f}$$



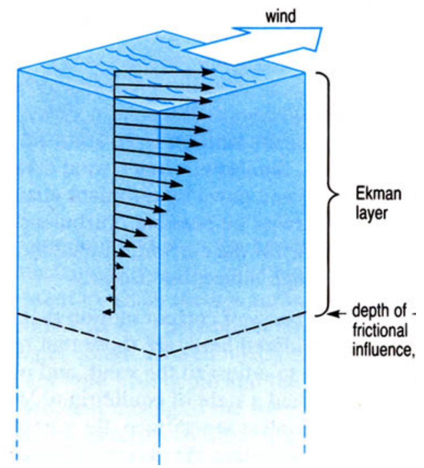
N.B. In Pinardi notes $U=M_{xe}$ and $V=M_{ye}$

With D_e (the Ekman "depth"): $D_e = \left(\frac{2A_v}{f\rho_0}\right)^{\frac{1}{2}}$

Whose solution gives transport at right angle with respect to wind Direction

Ekman equations and transport

Assuming $\tau_w^{(x)} = 0$ and $\tau_w^{(y)} \neq 0$ with y axis parallel to $\tau_w^{(y)}$ direction
The global problem reduces to:



$$\boxed{U = \frac{\tau_w^{(y)}}{f}}$$

$$\boxed{V = 0}$$